TITLE: Main Battle Tank Flexible Gun Tube Disturbance Model Three Segment Model

MAIN BATTLE TANK FLEXIBLE GUN TUBE DISTURBANCE MODEL THREE SEGMENT MODEL

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A rational approach to disturbance rejection is proposed and applied to a simple three-degree-of-freedom flexible gun tube model using feedforward and feedback compensation. The first two natural frequencies of the pin-free and cantilever tube are matched by adjusting the dimensions of the rigid segments and the stiffness of the torsional springs that join them. It was found that, contrary to the previously analyzed two degree-of-freedom segment model, the muzzle-end segment could be stabilized by the proper choice of transfer functions and elevation driveline response. The analysis serves to establish the requirements for the transfer functions and stabilizing actuator systems.

INTRODUCTION

Modern tank cannon are long, relatively, thin, beam-like hollow cylinders. Their accuracy is, in part, determined by their flexibility, especially under dynamic loading. Very small deflections and rotations of the muzzle end can have a significant influence on the accuracy of the shot at long ranges. Muzzle motions induced by firing are inevitable, and difficult to control because of the time scale of the firing is of the order of milliseconds.

Another source of muzzle motion is the ground-induced motion of the vehicle. These motions, transmitted through the trunnions and gun actuators, can be quite large and have frequencies comparable to the natural frequencies of the tube. The time scales of these disturbances depend on the tank speed and on the nature of the terrain. They are typically of the order of seconds or longer. Sensing and actuation to control the influence of vehicle motion on the muzzle response might be possible, given these relatively long time scales. This raises two questions. First, is it possible to reject some, or all he ground motion disturbance from the muzzle motion? In a previous paper [1] it was suggested that not all of the disturbance could be rejected. Second, if the more comprehensive model used here indicates that all of the disturbance can be rejected, what is the required control strategy?

During a discussion with Dr. Purdy, author of Reference [2], he suggested that the fidelity of his two-segment flexible model, documented in Reference [1], was inadequate. He recommended that the tube should be divided into at least three segments, with intervening

torsional springs and dampers. The author is indebted to Dr. Purdy for this suggestion since this paper is the result of his recommendation.

EQUATIONS OF MOTION

Figure 1 shows the generic model of the tube and the various quantities that determine its dynamic behavior.

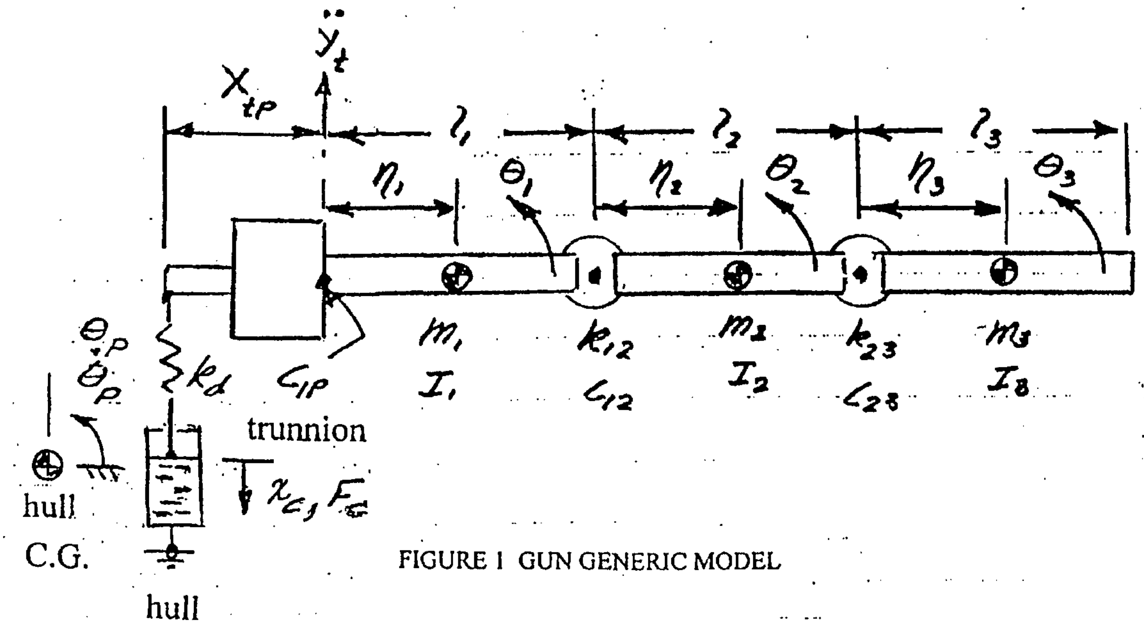

FIGURE 1 GUN GENERIC MODEL

A free body analysis of this model yields the classical dynamic equation,

$$[m]\{\ddot{\theta}\}+[c]\{\dot{\theta}\}+[k]\{\theta\}=\{I\} \tag{1}$$

The elements of the mass, damping, and stiffness 3 x 3 matrices are shown Appendix A.

Purdy [3] has shown that the tube motion can be adequately modeled if the segmented model matches the pinned-free and cantilever frequencies of the mounted tube. Matching is accomplished by adjusting the size of the rigid segments and the stiffness' of their connecting torsional springs. The 2 x 2 matrices for the cantilever mode are given in Appendix B.

Transformation of Equation (1) into the frequency domain will allow its incorporation into the control strategy. Taking the Laplace transform of Equation (1) yields:

$$[a]\begin{Bmatrix} x_c(s) \\ \theta_1(s) \\ \theta_2(s) \\ \theta_3(s) \end{Bmatrix} = [I]\begin{Bmatrix} F_c(s) \\ s^2 y_t(s) \\ s\theta_p(s) \\ \theta_p(s) \end{Bmatrix} \tag{2}$$

The elements of [a] and [I] are listed in Appendix C. This equation relates the response vector on the left to the disturbance vector on the right. These vectors also contain the actuator force, F_c, and the actuator displacement, x_c, in addition to disturbances ($s^2y_t(s)$, $s\theta_p(s)$, $\theta_p(s)$) and the responses $\theta_1(s)$, $\theta_2(s)$, $\theta_3(s)$.

$$\begin{Bmatrix} x_c(s) \\ \theta_1(s) \\ \theta_2(s) \\ \theta_3(s) \end{Bmatrix} = \frac{1}{\det[a]}[C]\,[I]\begin{Bmatrix} F_c(s) \\ s^2 y_t(s) \\ s\theta_p(s) \\ \theta_p(s) \end{Bmatrix} \tag{3}$$

where [C] is the transpose of the numerators of the cofactors of [a]. The elements of [C] and det [a] are listed in Appendix D.

The final step in the preparation of the dynamic equations is to perform the operation

$$\frac{1}{\det[a]}[C][I]=[B] \tag{4}$$

where the elements of [B] can be found in Appendix E.

The result of these straight forward, but laborious manipulations, is an equation for the response to the disturbance in terms of the properties of the model contained in [B], i.e.,

$$\begin{Bmatrix} x_c(s) \\ \theta_1(s) \\ \theta_2(s) \\ \theta_3(s) \end{Bmatrix} = [B]\begin{Bmatrix} F_c(s) \\ s^2 y_t(s) \\ s\theta_p(s) \\ \theta_p(s) \end{Bmatrix} \tag{5}$$

Of course it was known at the outset that Equation (1) could be put into this form. This section merely provides the details of how this transformation is performed, and documents the intermediate steps and their components.

FEEDBACK AND FEEDFORWARD CONTROL

The portion of the response due to the applied actuating force is

$$\begin{Bmatrix} x_c(s) \\ \theta_1(s) \\ \theta_2(s) \\ \theta_3(s) \end{Bmatrix} = \begin{bmatrix} B_{11} \\ B_{21} \\ B_{31} \\ B_{41} \end{bmatrix} F_c = [G_p] F_c \tag{6}$$

where $[G_p]$ is the "plant" transfer function.

The portion of the response due to the disturbance is

$$\begin{Bmatrix} \chi_c(s) \\ \theta_1(s) \\ \theta_2(s) \\ \theta_3(s) \end{Bmatrix}_p = \begin{bmatrix} B_{12} & B_{13} & B_{14} \\ B_{22} & B_{23} & B_{24} \\ B_{32} & B_{33} & B_{34} \\ B_{42} & B_{43} & B_{44} \end{bmatrix} \begin{Bmatrix} s^2 y_t(s) \\ s\theta_p(s) \\ \theta_p(s) \end{Bmatrix} = [G_d]\{D\} \tag{7}$$

where $[G_d]$ is the disturbance transfer function and $\{D\}$ is the disturbance vector.

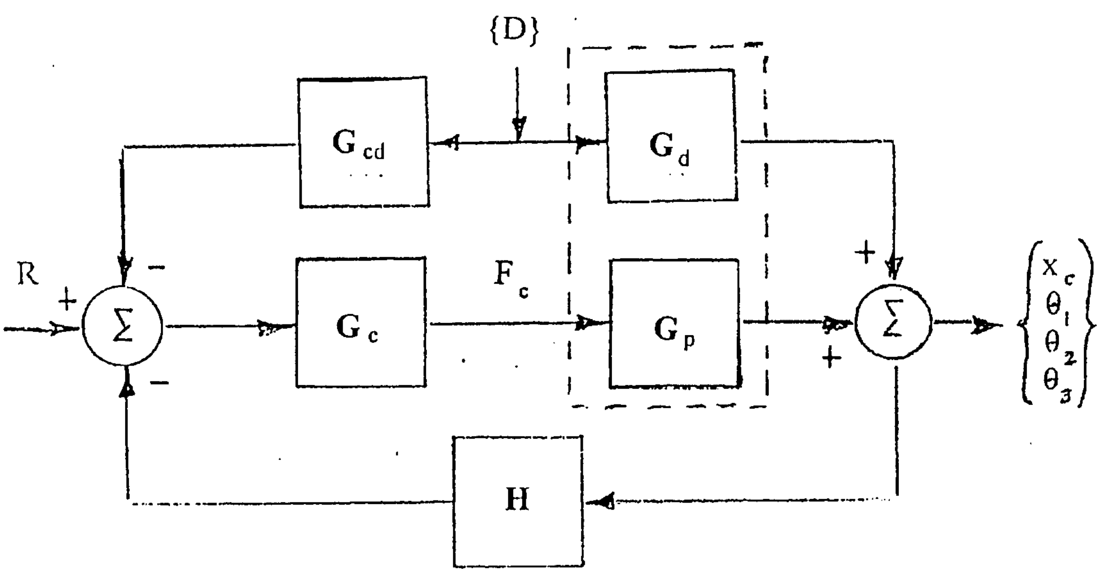

FIGURE 2 BLOCK DIAGRAM OF GUN TUBE SYSTEM

Figure 2 is a block diagram of the gun tube system with a gain G_c, feedback H, feedforward G_{cd}, and reference signal, R. Because R is a scalar the feedforward transfer function is a row vector, i.e.

$$[G_{cd}] = [G_{11}\ G_{12}\ G_{13}] \tag{8}$$

Assuming that the tube rotations at the trunnions and the muzzle can be sensed, the feedback transfer function is also a row vector, i.e.,

$$[H]=[0 \quad G_{22} \quad 0 \quad G_{24}] \tag{9}$$

Now referring to Figure 2 the response to the disturbance for the controlled system is

$$[[G_d]-[G_p][G_c][G_{cd}]]\{D\}=[[I]+[G_p][G_c][H]]\{\theta_D\} \tag{10a}$$

or in abbreviated notation

$$[d]\{D\}=[q]\{\theta_D\} \tag{10b}$$

The final step in these manipulations is to solve for the response to the disturbance, which is

$$\{\theta_D\}=[q]^{-1}[d]\{D\}=\frac{1}{\det[q]}[r]^T[d]\{D\} \tag{10c}$$

where [r] is the matrix of the cofactors of [q].

The matrices $[r]^T$ and [d] are given in Appendix F. It is interesting to note that [d] contains only the feedforward transfer functions G_{11}, G_{12}, and G_{13}, while [r] and det [q] contain only the feedback transfer functions, G_{22}, and G_{24}.

DISTURBANCE REJECTION

Referring to Appendix F the expanded version of Equation (10c) is

$$\begin{Bmatrix} x_c(s) \\ \theta_1(s) \\ \theta_2(s) \\ \theta_3(s) \end{Bmatrix}_D = \frac{1}{\det[q]} \begin{bmatrix} r_{11} & r_{21} & 0 & r_{41} \\ 0 & r_{22} & 0 & r_{42} \\ 0 & r_{23} & r_{33} & r_{43} \\ 0 & r_{24} & 0 & r_{44} \end{bmatrix} \begin{bmatrix} d_{11} & d_{12} & d_{13} \\ d_{21} & d_{22} & d_{23} \\ d_{31} & d_{32} & d_{33} \\ d_{41} & d_{42} & d_{43} \end{bmatrix} \begin{Bmatrix} s^2 y_t(s) \\ s\theta_p(s) \\ \theta_p(s) \end{Bmatrix} \tag{10d}$$

where

$$\det[q]=1+B_{21}\,G_c\,G_{22}+B_{41}\,G_c\,G_{24} \tag{11}$$

To completely remove the effect of the disturbances on $\theta_3(s)_D$ requires that

$$r_{24}\, d_{21} + r_{44}\, d_{41} = 0, \quad (12a)$$

$$r_{24}\, d_{22} + r_{44}\, d_{42} = 0, \quad (12b)$$

$$r_{24}\, d_{23} + r_{44}\, d_{43} = 0, \quad (12c)$$

and

$$\det[q] \neq 0 \quad (12d)$$

One way to accomplish this is to let $G_{22} = 0$ so that

$$r_{24} = -B_{41}\, G_c\, G_{22} = 0 \quad (13)$$

and then choose

$$d_{41} = B_{42} - B_{41}\, G_c\, G_{11} = 0, \quad (14a)$$

$$d_{42} = B_{43} - B_{41}\, G_c\, G_{12} = 0, \quad (14b)$$

$$d_{43} = B_{44} - B_{41}\, G_c\, G_{13} = 0 \quad (14c)$$

so that

$$\det[q] = 1 + B_{41}\, G_c\, G_{24} = r_{11} = r_{22} = r_{33}, \quad (14d)$$

and

$$r_{44} = 1\,,\, r_{21} = r_{23} = r_{24} = 0 \quad (15)$$

The effect of this choice on the disturbance transfer function is

$$\frac{1}{\det[q]}[r]^T\,[d] = \frac{1}{r_{11}} \begin{bmatrix} r_{11} & 0 & 0 & r_{41} \\ 0 & r_{22} & 0 & r_{42} \\ 0 & 0 & r_{33} & r_{43} \\ 0 & 0 & 0 & r_{44} \end{bmatrix} \begin{bmatrix} d_{11} & d_{12} & d_{13} \\ d_{21} & d_{22} & d_{23} \\ d_{31} & d_{32} & d_{33} \\ 0 & 0 & 0 \end{bmatrix} \quad (16)$$

An alternative strategy is to let $r_{44} = 1 + B_{21}\, G_c\, G_{22} = 0$, and then choose $d_{21} = d_{22} = d_{23} = 0$ so that $\det\,[q] = B_{41}\, G_c\, G_{24} = r_{11} = r_{33}$. However, in the end the resulting disturbance transfer function is the same as created by Eq (16).

Since the fourth column of $[r]^T$ is eliminated by the matrix multiplication and $\det[q] = r_{11} = r_{22} = r_{33}$, $\det[q]$ will be eliminated from the transformation.

IMPLEMENTATION

This analysis of the three-segment model indicates that model muzzle element can be stabilized by properly selecting the feedforward and feedback transfer functions. This is contrary to the finding for the lower order two-segment model [1]. Although the three-segment model only approximates the real tube, the results of this model are encouraging with respect to real tubes.

In order to achieve muzzle stabilization the breech-end of the gun must be actuated. Segments 1 and 2 will also rotate. These motions are determined by Equations (14) and Equation (16). All of the elements of [d] can be written in terms of the elements of [B]. Because $[r]^T$ acts like an identity matrix, the product $[r]^T[d]$ is quite simple. If the process is carried a step further the result can be put in terms of the elements of [C] with startling results, i.e., $d_{22} = d_{23} = d_{32} = d_{33} = 0$, and only $d_{11} = d_{12} = d_{13} = d_{21}$, and d_{31} are non-zero. The surviving elements are

$$d_{11} = \frac{1}{C_{14}\det[a]}\left((C_{14}C_{31} - C_{11}C_{34})I_{32} + (C_{14}C_{41} - C_{11}C_{44})I_{42}\right) \quad (17a)$$

$$d_{21} = \frac{1}{C_{14}\det[a]}\left((C_{14}C_{32} - C_{12}C_{34})I_{32} + (C_{14}C_{42} - C_{12}C_{44})I_{42}\right) \quad (17b)$$

$$d_{31} = \frac{1}{C_{14}\det[a]}\left((C_{14}C_{33} - C_{13}C_{34})I_{32} + (C_{14}C_{43} - C_{12}C_{44})I_{42}\right) \quad (17c)$$

$$d_{12} = \frac{1}{C_{14}\det[a]}(C_{14}C_{21} - C_{11}C_{24})I_{23} \quad (17d)$$

$$d_{13} = \frac{1}{C_{14}\det[a]}(C_{14}C_{21} - C_{11}C_{24})I_{24} \quad (17e)$$

These transfer functions relate the disturbances to the responses. All of the C's are of order s^4 with the exception of C_{11}, which is of order s^6. Since there is no restraining torsional spring connecting the tube to the mount in the model $s = 0$ is a root of $\det[a]$. Removing this rigid body factor from $\det[a]$ reduces it to order s^5.

TUBE MODEL PARAMETERS

The feedforward transfer functions depend on the length and mass properties of the segments, the torsional stiffness of the joining springs, and the torsional damping coefficients.

These are chosen so that the actual cantilever and pin-free mode shapes and natural frequencies are matched as closely as possible [3]. To simplify the matching process it is assumed that the damping is negligible. The first estimate of the segment lengths can be obtained by "fitting" the straight-line segments to the mode shapes obtained from a finite element model of the tube or other modal analyses. This fitting is best done by graphically overlaying the segments on plots of the mode shapes to estimate the segment lengths. The calculation of the mass properties of the segments can then be performed and these, along with the modal frequencies, inserted into the characteristic equations. The characteristic equations will then contain only the torsional stiffnesses as unknowns. The cantilever and pin-free equations are both quadratic so that the stiffness coefficients can be found directly. The degree of matching is determined by how close the cantilever and pin-free stiffnesses agree.

The characteristic equations for the cantilever and pin-free segments are given in Appendix G. Although the pin-free equation appears to be sixth-order it has a double root that is zero. The calculations for this trial-and-success process are easily implemented on a spreadsheet.

The XM291 tank gun was chosen for modeling because its mode shapes and frequencies were available from an existing, validated analytical model. Matching the stiffnesses proved to be surprisingly easy, requiring only modest adjustments to the first estimates of the segment lengths. Since all their frequencies (cantilever: 97.4 Hz, 40.35 Hz; pin-free: 25.08 Hz, 81.59 Hz) were inserted into the characteristic equations, they are matched exactly. The torsional stiffness for the pinned-free and cantilever modes were matched within 2% using the lengths l_1 = 6.0 ft, l_2 = 5.5 ft and l_3 = 6.0 ft. From this process the model torsional stiffness' $k_{12} = 3.6(10^6)$ lb ft/rad and $k_{23} = 1.69(10^6)$ lb ft/rad. were obtained

Dynamic analyses, [2], [3] have successfully modeled tube response using proportional damping, i.e. $[c] = \beta[k]$. In the case of the XM291 $\beta = 0.0015$ sec has been found to be reasonable. A reasonable estimate for trunnion damping is $c_{1p} = 750$ lb ft s/rad.

The first attempt to determine the feedforward transfer function using Equations (12) and (13) failed because the some of the roots of B_{41} were positive. This difficulty was eliminated by using the alternative strategy described above, with the following results.

$$r_{44} = 1 + B_{21}\, G_c\, G_{22} = 0 \tag{18}$$

$$G_c\, G_{11} = \frac{B_{22}}{B_{21}} \tag{19a}$$

$$G_c\, G_{12} = \frac{B_{23}}{B_{21}} \tag{19b}$$

$$G_c\, G_{13} = \frac{B_{24}}{B_{21}} \tag{19c}$$

The det [a] plays no role in these functions because it is canceled by ratioing the B's. Figures 3 and 4 show the Bode plots of $G_c G_{11}$ and $G_c G_{12}$. The transfer function $G_c G_{13}$ is zero so that θ_p is not fed forward. The numerators and denominators are all fifth order polynomials, so that the high and low frequency gains are bounded.

The feedback transfer function, $G_c G_{22}$, is shown in Figure 5. The remaining feedback transfer function, $G_c G_{24}$ plays no role in disturbance rejection in this case.

The elements of d_{22}, d_{23}, d_{32}, and d_{33} of [d] were found to be identically zero. The remaining no-zero elements of Equation (10d) and (16) yield the following response equations:

$$x_c = d_{11}\, s^2\, y_t(s) + d_{12}\, s\,\theta_p(s) + d_{13}\,\theta_p(s) \tag{20a}$$

$$\theta_1 = d_{21}\, s^2\, y_t(s) \tag{20b}$$

$$\theta_2 = d_{31} s^2 y_t(s) \tag{20c}$$

Figures 6 through 10 show the transfer functions required by the equations above. Figures 9 and 10 show that affect of the trunnion acceleration on θ_1 and θ_2 is highly attenuated so that large angular displacements of the tube are not required to achieve stabilization.

Figures 6 through 8 are quite similar. It appears that the required x_c will depend largely on the trunnion acceleration and pitch rate at very low frequencies. There is a considerable attenuation of the disturbance inputs up to 100 rad/sec (~15 Hz) with a return to the low frequency levels at 10^3 rad/sec (160 Hz).

CONCLUSIONS

It appears that the results previously obtained with the two-segment model, [1], led to the erroneous conclusion that the effects of the disturbance could not be entirely repeated from the muzzle angular displacement. The analysis of the three-segment model presented suggests that this is possible, at least theoretically. Of course the unanswered question is "what would be revealed by a higher order multi-segmented model, and how many segments are enough."

On the practical side, it is certain that the transfer functions cannot be duplicated precisely. There are four of these that must be implemented with reasonable fidelity to achieve the predicted results of the three-segment model. That number, along with their input signals, indicates the magnitude of the task. While feedforward and feedback control has long been used in fire control it is hoped that this paper provides some guidance in their use when tube flexure is a consideration.

NOMENCLATURE

[a] - dynamic matrix

[B] = [C] [I]

[c] = damping matrix

c_{12}, c_{23} = damping coefficients

c_{1p} = trunnion viscous friction coefficient

[C] = cofactor matrix of [a]

[d] = disturbance input matrix

{D} = disturbance vector

F_c = elevation actuating force

G_c = scalar gain

G_{cd} = feedforward transfer function vector

G_d = disturbance transfer function

G_p = plant transfer function

G_{11}, G_{12}, G_{13} = feedforward transfer function vector components

G_{22}, G_{24} = feedback transfer function vector components

H = feedback transfer function vector

[I] = forcing function matrix

[k] = stiffness matrix

k_{12}, k_{23} = stiffness coefficients

k_d = drive line stiffness

l_1, l_2, l_3 = segment lengths

[m] = mass matrix

[q] = disturbance response matrix

x_c = elevation actuator displacement

X_{tp} = distance from trunnions to drive

y_t = vertical displacement of the trunnion

[r] = cofactor matrix of [q]

β = proportional damping coefficient

η_1, η_2, η_3 = center of mass coordinates

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3. Purdy, D.J., "An Investigation into Modeling and Control Flexible Bodies", Ph.D. Thesis, Cranfield University, England, 1994.

APPENDIX A

$$[\mathbf{m}] = \begin{bmatrix} m_{11} & m_{12} & m_{13} \\ m_{21} & m_{22} & m_{23} \\ m_{31} & m_{32} & m_{33} \end{bmatrix}$$

$$m_{11} = I_1 + m_1\eta_1^2 + l_1^2(m_2 + m_3)$$
$$m_{22} = I_2 + m_2\eta_2^2 + m_3 l_2^2$$
$$m_{33} = I_3 + m_3\eta_3^2$$
$$m_{12} = m_{21} = m_2 l_1 \eta_2 + m_3 l_1 l_2$$
$$m_{13} = m_{31} = m_3 l_1 \eta_3$$
$$m_{23} = m_{32} = m_3 l_2 \eta_3$$

$$[\mathbf{c}] = \begin{bmatrix} c_{12} & -c_{12} & 0 \\ -c_{12} & c_{12} + c_{23} & -c_{23} \\ 0 & -c_{23} & c_{23} \end{bmatrix}$$

$$[\mathrm{k}] = \begin{bmatrix} k_{12} & -k_{12} & 0 \\ -k_{12} & k_{12} + k_{23} & -k_{23} \\ o & -k_{23} & k_{23} \end{bmatrix}$$

APPENDIX B

$$[m_c] = \begin{bmatrix} m_{22} & m_{23} \\ m_{32} & m_{33} \end{bmatrix} ; \quad [c_c] = \begin{bmatrix} c_{12} + c_{23} & -c_{23} \\ -c_{23} & c_{23} \end{bmatrix} ; \quad [k_c] = \begin{bmatrix} k_{12} + k_{23} & -k_{23} \\ -k_{23} & k_{23} \end{bmatrix}$$

APPENDIX C

$$[a] = \begin{bmatrix} a_{11} & a_{12} & 0 & 0 \\ a_{21} & a_{22} & a_{23} & a_{24} \\ 0 & a_{32} & a_{33} & a_{34} \\ 0 & a_{42} & a_{43} & a_{44} \end{bmatrix}$$

$$a_{11} = k_d$$
$$a_{12} = a_{21} = -k_d X_{tp}$$
$$a_{24} = a_{42} = m_{13} s^2$$

$$a_{22} = m_{11}s^2 + (c_{12} + c_{1p})s + (k_d X_{tp}{}^2 + k_{12})$$
$$a_{23} = a_{32} = m_{12}s^2 - c_{12}s - k_{12}$$
$$a_{33} = m_{22}s^2 + (c_{12} + c_{23})s + (k_{12} + k_{23})$$
$$a_{34} = a_{43} = m_{23}s^2 - c_{23}s - k_{23}$$
$$a_{44} = m_{33}s^2 + c_{23}s + k_{23}$$

$$[I]=\begin{bmatrix} I_{11} & 0 & 0 & I_{14} \\ 0 & I_{22} & I_{23} & I_{24} \\ 0 & I_{32} & 0 & 0 \\ 0 & I_{42} & 0 & 0 \end{bmatrix}$$

$$I_{11} = 1.0$$
$$I_{14} = -k_d X_{tp}$$

$$I_{22} = -(\eta_1 m_1 + l_1 m_2 + l_1 m_3)$$
$$I_{23} = c_{1p}$$
$$I_{24} = k_d X_{tp}^{\ 2}$$
$$I_{32} = -(\eta_2 m_2 + l_2 m_3)$$
$$I_{42} = -\eta_3 m_3$$

APPENDIX D

$$[C]=\begin{bmatrix} C_{11} & C_{12} & C_{13} & C_{14} \\ C_{21} & C_{22} & C_{23} & C_{24} \\ C_{31} & C_{32} & C_{33} & C_{34} \\ C_{41} & C_{42} & C_{43} & C_{44} \end{bmatrix}$$

$$C_{11} = a_{22}a_{33}a_{44} + 2a_{23}a_{34}a_{24} - (a_{24}a_{33}a_{42} + a_{43}a_{34}a_{22} + a_{23}a_{32}a_{44})$$
$$C_{22} = a_{11}(a_{33}a_{44} - a_{34}a_{43})$$
$$C_{33} = a_{11}(a_{22}a_{44} - a_{24}a_{42})$$
$$C_{44} = a_{11}(a_{22}a_{33} - a_{23}a_{32}) - a_{12}a_{21}a_{33}$$
$$C_{12} = C_{21} = -a_{21}(a_{33}a_{44} - a_{34}a_{43})$$
$$C_{13} = C_{31} = a_{21}(a_{33}a_{44} - a_{34}a_{42})$$
$$C_{14} = C_{41} = -a_{21}(a_{32}a_{43} - a_{33}a_{42})$$
$$C_{23} = C_{32} = -a_{11}(a_{32}a_{44} - a_{34}a_{42})$$

$$C_{24} = C_{42} = a_{11}(a_{32}a_{43} - a_{33}a_{42})$$
$$C_{34} = C_{43} = -a_{11}(a_{22}a_{43} - a_{23}a_{42}) + a_{12}a_{21}a_{43}$$

$$\det[a] = a_{11}(a_{22}a_{33}a_{44} + a_{23}a_{34}a_{42} + a_{32}a_{43}a_{24})$$
$$- a_{11}(a_{24}a_{33}a_{42} + a_{34}a_{43}a_{22} + a_{23}a_{32}a_{44}) - a_{12}a_{21}(a_{33}a_{44} - a_{34}a_{43})$$

APPENDIX E

$$[B]=\begin{bmatrix} B_{11} & B_{12} & B_{13} & B_{14} \\ B_{21} & B_{22} & B_{23} & B_{24} \\ B_{31} & B_{32} & B_{33} & B_{34} \\ B_{41} & B_{42} & B_{43} & B_{44} \end{bmatrix}$$

$$B_{11} = C_{11}I_{11} / \det[a]$$
$$B_{12} = (C_{21}I_{22} + C_{31}I_{32} + C_{41}I_{42}) / \det[a]$$
$$B_{13} = (C_{21}I_{23}) / \det[a]$$
$$B_{14} = (C_{11}I_{14} + C_{21}I_{24}) / \det[a]$$
$$B_{21} + (C_{12}I_{11}) / \det[a]$$
$$B_{22} = (C_{22}I_{22} + C_{32}I_{32} + C_{42}I_{42}) / \det[a]$$
$$B_{23} = (C_{22}I_{23}) / \det[a]$$
$$B_{24} = (C_{12}I_{14} + C_{22}I_{24}) / \det[a]$$
$$B_{31} = (C_{13}I_{11}) / \det[a]$$
$$B_{32} = (C_{23}I_{22} + C_{33}I_{32} + C_{43}I_{42}) / \det[a]$$
$$B_{33} = (C_{23}I_{23}) / \det[a]$$
$$B_{34} = (C_{13}I_{14} + C_{23}I_{24}) / \det[a]$$
$$B_{41} = (C_{14}I_{11}) / \det[a]$$
$$B_{42} = (C_{24}I_{22} + C_{34}I_{32} + C_{44}I_{42}) / \det[a]$$
$$B_{43} = (C_{24}I_{23}) / \det[a]$$
$$B_{44} = (C_{14}I_{14} + C_{24}I_{24}) / \det[a]$$

APPENDIX F

$$[r]^T = \begin{bmatrix} r_{11} & r_{21} & 0 & r_{41} \\ 0 & r_{22} & 0 & r_{42} \\ 0 & r_{23} & r_{33} & r_{43} \\ 0 & r_{24} & 0 & r_{44} \end{bmatrix}$$

$$r_{11} = \det[a] = 1 + B_{21}G_cG_{22} + B_{41}G_cG_{24} \qquad r_{33} = r_{11}$$
$$r_{21} = -B_{11}G_cG_{22} \qquad r_{41} = -B_{11}G_cG_{24}$$
$$r_{22} = 1 + B_{41}G_cG_{24} \qquad r_{42} = -B_{21}G_cG_{24}$$
$$r_{23} = -B_{31}G_cG_{22} \qquad r_{43} = -B_{31}G_cG_{24}$$
$$r_{24} = -B_{41}G_cG_{22} \qquad r_{44} = 1 + B_{21}G_cG_{22}$$

$$[d]=\begin{bmatrix} d_{11} & d_{12} & d_{13} \\ d_{21} & d_{22} & d_{23} \\ d_{31} & d_{32} & d_{33} \\ d_{41} & d_{42} & d_{43} \end{bmatrix}$$

$$d_{11} = B_{12} - B_{11}G_cG_{11}$$
$$d_{21} = B_{22} - B_{21}G_cG_{11}$$
$$d_{31} = B_{32} - B_{31}G_cG_{11}$$
$$d_{41} = B_{42} - B_{41}G_cG_{11}$$
$$d_{12} = B_{13} - B_{11}G_cG_{12}$$
$$d_{22} = B_{23} - B_{21}G_cG_{12}$$
$$d_{32} = B_{33} - B_{31}G_cG_{12}$$
$$d_{42} = B_{43} - B_{41}G_cG_{12}$$
$$d_{13} = B_{14} - B_{11}G_cG_{13}$$
$$d_{23} = B_{24} - B_{21}G_cG_{13}$$
$$d_{33} = B_{34} - B_{31}G_cG_{13}$$
$$d_{43} = B_{44} - B_{41}G_cG_{13}$$

APPENDIX G

$$\varpi_1 = \varpi_2 = \textit{first two natural frequencies}$$

$$k_{23}^2 - \left[\left(\varpi_1^2 + \varpi_2^2\right)\left(m_{22}m_{33} - m_{23}m_{32}\right)/\left(m_{33} + 2m_{23} + m_{22}\right)\right]k_{23}$$
$$+ m_{33}\left(\varpi_1^2\varpi_2^2\right)\left(m_{22}m_{33} - m_{23}m_{32}\right)/\left(m_{33} + 2m_{23} + m_{22}\right) = 0$$

$$k_{12} = \left(\varpi_1^2 + \varpi_2^2\right)\left(m_{22}m_{33} - m_{23}m_{32}\right)/m_{33} - \left(m_{33} + 2m_{23} + m_{22}\right)k_{23}/m_{33}$$